NEGATIVE AND CONTINUOUS LOTTERIES IN THE HALF-FULL/HALF-EMPTY FRAMEWORK

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Abstract

In the paper Cenci *et al*. (2015) an intuitive and parsimonious descriptive model of decision making under risk that can explain the Kahneman and Tversky's (1979) paradoxes has been proposed for non-negative lotteries. In this paper it is shown that a straightforward extension of the value function proposed in Cenci *et al*. (2015) explains the Kahneman and Tversky's (1979) paradoxes also for negative lotteries. Starting from the value function for discrete lotteries the value function for a continuous lottery is obtained.

Keywords: Decision under risk, Decision-making paradoxes, Optimism/pessimism, Expected value criterion, Prospect Theory.

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1 Introduction

The literature on decision-making under risk has been driven by the discovery of paradoxes afflicting existing theories, such as expected value (EV) criterion and expected utility (EU) theory. (See, for instances, Fox, Erner, & Walters, in press, Luce, Ng, Marley, and Aczél (2008), Wu, Zhang, & Gonzalez (2004)). Paradoxes such as the Allais (1953) paradox and its variations (Kahneman and Tversky (1979)) challenge the *descriptive* power of the expected value (EV) criterion and expected utility (EU) theory. For this reason decision theorists have sought a theory of choice able to provide a satisfactory *description* of decision-makers' behavior in risky contexts.

Nowadays Kahneman and Tversky's (1979) original Prospect Theory and Tversky and Kahneman (1992) New Prospect Theory have become the leading descriptive framework for modeling decision making under risk (Camerer (1998), Fox and Poldrack (2014)).

In the recent paper, Cenci *et al*. (2015), an intuitive and parsimonious descriptive model of decision making under risk that can explain the Kahneman and Tversky's (1979) paradoxes has been proposed for non-negative lotteries. The intuition underlying this model is the metaphor of the ‘‘glass half-full/glass half-empty’’, which suggests that different individuals may evaluate the same situation in different ways (psychological effects/cognitive bias), and will typically be divided between those who focus on positive aspects of the situation (the optimists, glass half-full) and those who focus on negative aspects (the pessimists, glass half-empty). Cenci *et al*. (2015) suggests that a decision maker values the difference between the outcomes above (glass half-full) and below (glass half-empty) the mean value of the given lottery. In the model proposed in Cenci *et al*. (2015) the lottery's evaluation (and consequently the decision maker's risk attitude) depends only on two parameters: the degree of optimism/pessimism λ, (here λ_+) and on the probability distortion parameter q that determines decision weights. Cenci *et al*. (2015) shows that λ and q can be calibrated in such a way that Kahneman and Tversky's (1979) paradoxes are explained.

In section 2 the main results obtained in Cenci *et al*. (2015), henceforth $HFHE$ model, are summarized. In section 3 the results obtained in Cenci *et al*. (2015) are extended to negative lotteries. The paradoxes in Kahneman and Tversky's (1979) involving negative lotteries are simply explained considering a straightforward generalization of the $HFHE$ value function. In section 4 the $HFHE$ value function for a continuous lottery is obtained. Section 5 close with a shortly conclusion.

2 The HFHE model

The glass half-full/half-empty adage suggests that, when observing a glass that contains 50% of water, optimists focus on the half-full part and therefore see the glass half-full, while pessimists focus on the half-empty part and therefore see the glass half-empty. The decisions maker may also distort the lottery's probabilities, something widely recognized in the literature (see, for instance, Fehr-Duda and Epper (2012), Kahneman & Tversky (1979), and Tversky and Wakker (1995)).

This intuition has been used in Cenci *et al*. (2015) in order to construct a value function for a lottery with non-negative outcomes.

Considering a lottery $X = \{(x_1, p_1); (x_2, p_2); \dots; (x_n, p_n)\}$ with $x_i \geq 0$, for $i = 1, \dots, n$, the value function ($HFHE$ model) is obtained as

$$H(X) = \mu_q + 2(2\lambda_+ - 1)E_q[(X - \mu_q)_+], \tag{1}$$

where

- $\mu_q = \frac{\sum_{i=1}^{n}(p_i{}^q)x_i}{\sum_{j=1}^{n} p_j{}^q}$
- $(a - b)_+ = a - b \; if \; a \geq b$
- $(a - b)_+ = 0 \; if \; a < b$ (2)
- $E_q[g(X)] = \frac{\sum_{i=1}^{n}(p_i{}^q)g(x_i)}{\sum_{j=1}^{n} p_j{}^q}$

The value function HFHE in Eq.1 explain, as shown in Cenci *et al*. (2015), all the paradoxes with positive outcomes proposed in Kahneman and Tversky (1979), provided that parameters are as in Table 1.

q	λ_+
0.44	(0.25,0.31)
0.46	(0.25,0.32)
0.48	(0.25,0.33)
0.50	(0.25,0.34)
0.52	(0.25,0.35)

Table 1

The selected parameters in Cenci *et al*. (2015).

3 Negative lotteries

For negative lotteries,

$$X = \{(x_1, p_1); (x_2, p_2); \dots; (x_n, p_n)\}$$

with $x_i \leq 0$, the value functions

$$H(X) = \mu_q + 2(2\lambda_- - 1)E_q[(X - \mu_q)_+] \quad (3)$$

is proposed, where μ_q, $(a-b)_+$ and $E_q[g(X)]$ are as in Eq.2.

The value function in Eq.3 can be used to investigates the paradoxes in Kahneman and Tversky (1979) for negative lotteries. Table 2 summarizes the results obtained.

q	P3'	P4'	P7'	P8'	P12	P13'	P14'	λ_-
0.44	$\lambda_- > 0.39$	$\lambda_- < 0.82$	$\lambda_- > 0.69$	$\lambda_- < 0.75$	$\lambda_- > 0.50$	$\lambda_- > 0.68$	$\lambda_- < 0.76$	(0.69,0.75)
0.46	$\lambda_- > 0.39$	$\lambda_- < 0.82$	$\lambda_- > 0.68$	$\lambda_- < 0.75$	$\lambda_- > 0.50$	$\lambda_- > 0.67$	$\lambda_- < 0.75$	(0.68,0.75)
0.48	$\lambda_- > 0.40$	$\lambda_- < 0.81$	$\lambda_- > 0.67$	$\lambda_- < 0.75$	$\lambda_- > 0.50$	$\lambda_- > 0.67$	$\lambda_- < 0.75$	(0.67,0.75)
0.50	$\lambda_- > 0.41$	$\lambda_- < 0.80$	$\lambda_- > 0.66$	$\lambda_- < 0.75$	$\lambda_- > 0.50$	$\lambda_- > 0.66$	$\lambda_- < 0.75$	(0.66,0.75)
0.52	$\lambda_- > 0.41$	$\lambda_- < 0.80$	$\lambda_- > 0.65$	$\lambda_- < 0.75$	$\lambda_- > 0.50$	$\lambda_- > 0.66$	$\lambda_- < 0.75$	(0.66,0.75)

Table 2

q and λ_-values explaining Kahneman and Tversky (1979) paradoxes for negative lotteries. PX is short for Problem X in Kahneman and Tversky (1979).

Results reported in Table 2 show that the value function in Eq.3 explain the paradoxes in Kahneman and Tversky (1979) for negative lotteries: for a given q in the first column the corresponding interval of λ_- values that explains all the paradoxes in Kahneman and Tversky (1979) for negative lotteries is given in the last column.

Remark

From the results reported in Table 1 and Table 2 it follows that, for a given q, the relation

$$\lambda_- = 1 - \lambda_+$$

holds.

4 Continuous lottery

Let X be a discrete random variable, given by $X = \{(x_i, p_i)\}$, $i = 1, \ldots, N$, with $x_i < x_{i+1}$ and $x_1 = a, x_N = b$.

Definition

The discrete random variable X is said *regular* if there exists a Riemann integrable function

$$\rho: [\mathrm{a}, \mathrm{b}] \to \mathbb{R}$$

such that

$$p_i = \rho(x_i)\Delta x_N, \tag{4}$$

where

$$\Delta x_N = max_{i=1,\ldots,N}\{x_i - x_{i-1}\}$$

and

$$\lim_{N\to+\infty} \Delta x_N = 0.$$

Proposition 1

Let $g(x)$ be a Riemann integrable function in $[\mathrm{a}, \mathrm{b}]$ and $X = \{(x_i, p_i)\}$, $i = 1, \ldots, N$, with $x_i < x_{i+1}$ and $x_1 = a, x_N = b$ a regular discrete random variable.

It follows that

$$\lim_{N\to+\infty} \sum_{i=1}^{N} g(x_i)\Delta x_N = \int_a^b g(x)dx \ .$$

Proof

Since $g(x)$ is a Riemann integrable function in $[\mathrm{a}, \mathrm{b}]$ it follows that

$$\int_a^b g(x)dx = \lim_{N\to+\infty} \sum_{i=1}^{N} g({x_i}^*)\Delta x_{i,N}$$

for an arbitrary partition P^* of the interval $[\mathrm{a}, \mathrm{b}]$ such that

$$P^*: a = {x_1}^* < {x_2}^* <\ldots< {x_N}^* = b$$

with ${x_{i+1}}^* = {x_i}^* + \Delta x_{i,N}$, $i = 1, \ldots N-1$ and $\lim_{N\to+\infty} \Delta x_{i,N} = 0$.

The choice of the particular partition

$$P \ : a = x_1 \ < x_2 \ <\ldots< x_N \ = b$$

with $x_{i+1} \ = x_i \ + \Delta x_N$, $i = 1,\ldots N-1$ and $\lim_{N\to+\infty} \Delta x_N = 0$

shows the thesis validity. ■

Proposition 2

Let $X = \{(x_i, p_i)\}, i = 1,\ldots,N$, with $x_i < x_{i+1}$ and $x_1 = a, x_N = b$ be a regular discrete random variable.

Let

$$w(p_i) = \frac{{p_i}^q}{\sum_{j=1}^N {p_j}^q} \ , i = 1,\ldots,N$$

be a probability distortion function, and let

$$\phi\colon [\mathrm{a},\mathrm{b}] \to \ \mathbb{R}$$

be a Riemann integrable function.

It follows that

$$\lim_{N\to+\infty} \sum_{i=1}^{N} w(p_i)\phi(x_i) = \int_a^b w(\rho(\mathrm{x}))\phi(\mathrm{x})dx \ ,$$

where

$$w(\rho(\mathrm{x})) = \frac{\rho(\mathrm{x})^q}{\int_a^b \rho(\mathrm{y})^q dy} \ ,$$

and $\rho(\mathrm{x})$ is given by Eq.4.

Proof

We have

$$\sum_{i=1}^N w(p_i)\phi(x_i) = \frac{\sum_{i=1}^N {p_i}^q \phi(x_i)}{\sum_{j=1}^N {p_j}^q} \ = \qquad (5)$$

$$= \frac{\sum_{i=1}^N \rho(x_i)^q \phi(x_i) {\Delta x_N}^q}{\sum_{j=1}^N \rho(x_j)^q {\Delta x_N}^q} = \frac{\sum_{i=1}^N \rho(x_i)^q \phi(x_i) \Delta x_N}{\sum_{j=1}^N \rho(x_j)^q \Delta x_N} \qquad (6)$$

Using Proposition 1 we obtain from Eq.5 and Eq.6

$$\lim_{N\to+\infty}\sum_{i=1}^{N} w(p_i)\phi(x_i) = \lim_{N\to+\infty}\frac{\sum_{i=1}^{N}\rho(x_i)^q\phi(x_i)\Delta x_N}{\sum_{j=1}^{N}\rho(x_j)^q\Delta x_N} =$$

$$= \frac{\int_a^b \rho(\mathrm{x})^q\,\phi(\mathrm{x})\,\mathrm{dx}}{\int_a^b \rho(\mathrm{y})^q\,\mathrm{dy}} = \int_a^b w(\rho(\mathrm{x}))\phi(\mathrm{x})dx\,.$$

■

Remark

Proposition 2 shows that for a continuous random variable the value function

$$H(X) = \mu_q + 2(2\lambda - 1)E_q\left[\left(X - \mu_q\right)_+\right]$$

where

- $\mu_q = \frac{\int_a^b \rho(\mathrm{x})^q \mathrm{x}\,\mathrm{dx}}{\int_a^b \rho(\mathrm{y})^q\,\mathrm{dy}}$
- $(a-b)_+ = a - b \text{ } if \text{ } a \geq b$
- $(a-b)_+ = 0 \text{ } if \text{ } a < b$ (2)
- $E_q[g(X)] = \frac{\int_a^b \rho(\mathrm{x})^q \mathrm{g}(\mathrm{x})\,\mathrm{dx}}{\int_a^b \rho(\mathrm{y})^q\,\mathrm{dy}}$

can be used as value function for continuous lotteries.

5 Conclusion

In this paper the results obtained in Cenci *et al*. (2015), an intuitive and parsimonious descriptive model of decision making under risk that can explain the Kahneman and Tversky's (1979) paradoxes for non-negative lotteries, are considered as starting point. We show that a straightforward extension of the value function proposed in Cenci *et al*. (2015) is able to explain the Kahneman and Tversky's (1979) paradoxes also for negative lotteries. The value function for a continuous lottery has been obtained as simple generalization of the value function for discrete lotteries.

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